Technical Report No. 32-447

A Technique for Optimum Final Value Control of Powered Flight Trajectories

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jpl

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The linear perturbation equations describing the first variation of the state variables along a powered flight path are employed to develop the necessary conditions which must be satisfied on an optimized standard trajectory. It is shown that the minimization of some explicit function of the end coordinates, subject to certain other boundary functions being zero, can be interpreted as being the limiting case of minimizing the sum of the squares of variations in these functions. Employing this interpretation, a control law is developed to obtain neighboring optimum trajectories in the presence of small initial condition disturbances. The variations in the boundary functions resulting from this control law are derived, and the effect of varying the final time is discussed. It is shown that the system is always stable, and is "in the limit" controllable. The technique is applied to the control of satellite orbit injection.

A TECHNIQUE FOR OPTIMUM FINAL VALUE CONTROL OF POWERED FLIGHT TRAJECTORIES

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I. INTRODUCTION AND SUMMARY

This paper develops a technique for controlling the acceleration vector along a powered flight trajectory, the scheme being based upon considering linear perturbations about a standard (nominal) trajectory. It is assumed that the standard trajectory has been optimized by chosing a control program which, for the given initial condition, causes the final state to be attained in such a way as to minimize an explicit function of the end coordinates and time, subject to certain end constraints. The task, then, is to construct a perturbed control program, to be applied between the initial and final time, which will re-optimize the trajectory in the presence of small disturbances to the initial conditions.

This well-known problem is complicated by the optimality property of the standard trajectory, which implies that variations in the control program have no first-order effect on the pay-off function. Recent work by Breakwell, Bryson, and Kelly (Ref. 1 and 2) considers the second variation of the standard trajectory, leading to a rather complex control law which possesses potential instability near the end point. This difficulty is eliminated here by interpreting the optimization problem to be "equivalent" (in a first-order sense) to the minimization of a certain quadratic function of the end coordinates.

Employing this interpretation in constructing neighboring extremals, it is shown that a simple and well-behaved control law results from ignoring the second variation of the state variables from the standard trajectory, thereby developing control equations which are functions of the elements of the familiar state-transition matrices (the adjoint variables). This approach effectively yields a trajectory which achieves the

standard end conditions "as closely as possible". Thus if $\delta \bar{q}_f$ is a vector composed of variations in the "boundary functions" defined at the fixed final time t_f, the corrective control between fixed initial and final times yields

$$\delta \bar{q}_f = k [kI + W]^{-1} \delta \bar{q}_0$$

where $\delta \bar{q}_0$ is evaluated from initial condition variations, W is a known matrix, I is the identity, and k is the "uncorrectable" component of $\delta \bar{q}_f$. The variation resulting from small changes in the final time is derived, which is then combined with $\delta \bar{q}_f$ to yield the total $\delta \bar{q}$. The "controllability" of the system is discussed, and it is shown how the important velocity-to-be gained approach to guidance and control problems can be placed within the framework of this analysis. An application of the method is presented.

II. DESCRIPTION OF THE PROBLEM[1]

A point mass is moving under controlled acceleration according to

$$\dot{\bar{x}} = \bar{f}(\bar{x}, \bar{c}, t) \tag{1}$$

where the x_i are the n state variables of the system (the position and velocity coordinates), the c_i are the m independent control variables of the system (such as the attitude angles of the control acceleration vector), and t is time. The functions f_i are assumed to be piecewise differentiable with respect to the state and control variables, and the control variables may be bounded. There exists an optimized standard (nominal) trajectory which, starting from the initial condition $\bar{x}_s(t_0)$, and employing the control variable program $\bar{c}_s(t)$, attains the final state $\bar{x}_s(t_f)$ in such a way as to minimize at t_f the function $p_1(\bar{x}, t)$, subject to the constraints $p_i(\bar{x}, t) = $ given,[2] where $i = 2, \cdots, l$. Without loss of generality, all p_{is} will be assumed equal to zero. It follows that the "penalty" function minimized at t_f on the standard trajectory is (Ref. 3)

$$p = \bar{v}^T \bar{p} = \sum_{i=1}^{l} v_i p_i(\bar{x}, t_f) \tag{2}$$

---

where the ν_i are known constants, ν_1 being taken equal to 1.0. (The ν_i must be found by an iterative search procedure in the construction of the standard trajectory. Examples of optimum powered flight trajectories are discussed in Ref. 4.) Control is terminated (control acceleration set equal to zero) at t_f. It is the purpose of this paper to develop a perturbed control variable program, $\delta \bar{c}(t) = \bar{c}(t) - \bar{c}_s(t)$ for $t_0 \leq t \leq t_f$, to be employed in the presence of a small initial condition variation, $\delta \bar{x}(t_0)$, which will cause the standard end conditions to be attained "as closely as possible". The technique for doing this will be discussed in Sec. IV.

Following a useful approach of guidance and control analysis (Ref. 5), the first variation of the state variables from the standard trajectory is described by the "variational equation"

$$\frac{d}{dt}(\delta \bar{x}) = F \delta \bar{x} + G \delta \bar{c} \tag{3}$$

where $\delta \bar{x} = \bar{x}(t) - \bar{x}_s(t)$, $\delta \bar{c} = \bar{c}(t) - \bar{c}_s(t)$, G is the $n \times m$ partial derivative matrix

$$\left[\frac{\partial \bar{f}(t)}{\partial \bar{c}(t)} \right]$$

and F is then $n \times n$ partial derivative matrix

$$\left[\frac{\partial \bar{f}(t)}{\partial \bar{x}(t)} \right]$$

Associated with Eq. (3) is the state transition matrix $U(t_f, t)$ which, along the standard trajectory, follows the differential equation

$$\frac{d}{dt} U + UF = 0 \tag{4}$$

and has end conditions

$$U(t_f, t_f) = I \tag{5}$$

where I is the identity matrix, and t_f is a given final time. The elements of U are the familiar adjoint variables. Thus

$$\frac{d}{dt}[U \delta \bar{x}] = U G \delta \bar{c} \tag{6}$$

and

$$\delta \overline{x}_f = \delta \overline{x}(t_f) = U(t_f, t_0) \, \delta \overline{x}(t_0) + \int_{t_0}^{t_f} U(t_f, t) \, G(t) \, \delta \overline{c}(t) \, dt \tag{7}$$

Equation (7) is the fundamental equation of linear control analysis. It will be the linearized problem described by Eq. (7) which will be considered in this paper, assuming that the U matrix is invariant in a small neighborhood of the standard trajectory.

III.   THE OPTIMALITY CONDITIONS

Equation (7) leads to a derivation of necessary conditions that must be satisfied in order for a trajectory to be a minimum with respect to the control variables, in particular, the conditions satisfied on the standard trajectory (Ref. 6).

From Eq. (2), the first variation in p is

$$\delta p = \overline{\nu}^T P \, \delta \overline{x}_f \tag{8}$$

where P is the $l \times n$ matrix $[\partial \overline{p} / \partial \overline{x}_f]$. From Eq. (7), the first variation due to $\delta \overline{c}(t)$ along the path is

$$\delta p = \overline{\nu}^T P \int_{t_0}^{t_f} U \, G \, \delta \overline{c} \, dt = \int_{t_0}^{t_f} \overline{\eta}^T \, \delta \overline{c} \, dt \tag{9}$$

where

$$\overline{\eta}^T = \overline{\nu}^T P \, U \, G \tag{10}$$

Let $\delta \overline{c}(t)$ be allowed to range between the boundaries

$$c_i \min(t) - c_{is}(t) \leq \delta c_i(t) \leq c_i \max(t) - c_{is}(t) \tag{11}$$

Consider an allowable variation $\delta \overline{c}(t) \neq 0$, which is applied for an interval ϵ, and suppose that the influence function $\eta_i \neq 0$ in this interval. Then the total variation in p due to $\delta c_i(t)$ is

$$\triangle p = \int_\epsilon \eta_i \, \delta c_i \, dt + \text{higher order terms} \tag{12}$$

By taking $\delta c_i(t) = $ constant, it follows that

$$\lim_{\epsilon \to 0} \triangle p = \eta_i(t) \, \delta c_i(t) \, \epsilon \geq 0 \tag{13}$$

if p is, in fact, minimized on the standard trajectory. This implies that c_{is} must be maximum if $\eta_i < 0$, or minimum if $\eta_i > 0$. Suppose instead that the $\overline{\eta}(t) = 0$ in the interval ϵ, causing p to be stationary with respect to the control variables. Considering the second variation of p, it follows that

$$\lim_{\epsilon \to 0} \Delta p = \delta \overline{c}^T(t) D(t) \delta \overline{c}(t) \epsilon \geq 0 \qquad (14)$$

where

$$D = \left[\frac{\partial \overline{\eta}(t)}{\partial \overline{c}(t)} \right] \qquad (15)$$

This implies that the matrix D is positive semi-definite over the interval ϵ, which is a form of the classical Legendre necessary condition. Since the c_i are assumed independent, Eq. (13) and (14) can be combined to state necessary conditions which must be satisfied on trajectories in the neighborhood of the path which yields minimum p: either the control is on the boundary or

$$\overline{\eta}(t) = 0 \qquad \text{and} \qquad \delta \overline{c}^T(t) D(t) \delta \overline{c}(t) \geq 0 \qquad (16)$$

where $\delta \overline{c}(t)$ is an arbitrarily small variation as allowed by Eq. (11). Equation (16) is analogous to the necessary condition for a local minimum in the ordinary calculus.

The influence functions $\eta_i(t)$ are evaluated on the optimum trajectory, which must be found, in general, by numerical search procedures. If it is assumed, however, that this task has already been accomplished in constructing the standard trajectory, and that U matrix elements (Eq. 4) do not change on trajectories near by the standard, the optimum control on perturbed trajectories is attained by either setting c_i equal to one of its bounding values, or else applying Eq. (10) to achieve optimality with respect to c_i. The later situation is the most interesting, and will be the subject of the remainder of this paper.

The optimality conditions developed above can be concisely summarized as follows. Define[3]

$$\overline{\lambda}^T(t) = \overline{\nu}^T P U(t_f, t) \qquad (17)$$

$$h(t) = \overline{\lambda}^T \overline{f}(\overline{x}, \overline{c}, t) \qquad (18)$$

---

[3] The $\lambda_i(t)$ are called differential corrections for the function $p(\overline{x}, t_f)$.

where, from Eq. (1) and (4)

$$\dot{x}_i = \left(\frac{\partial h}{\partial \lambda_i} \right) \tag{19}$$

$$\dot{\lambda}_i = -\left(\frac{\partial h}{\partial x_i} \right) \tag{20}$$

and, since $U(t_f, t_f) = I$, the boundary conditions on $\lambda_i(t_f)$ are

$$\lambda_i(t_f) = \left(\frac{\partial p}{\partial x_i} \right) + \sum_{j=2}^{l} \nu_j \left(\frac{\partial p_j}{\partial x_i} \right) \tag{21}$$

Then the Pontryagin maximum principle, or generalized Weierstrass condition, states that $-h$ is maximum with respect to the control variables (Ref. 7).

IV.   THE MINIMIZED QUADRATIC FORM

Let the standard trajectory be stationary with respect to the control variables, and consider the problem of determining the control $\delta \bar{c}(t)$ which re-optimizes the trajectory for small changes in the initial state, $\delta \bar{x}(t_0)$. Since $\delta \bar{c}(t)$ has no first-order effect on the return function p, it is necessary to consider at least the second variation of $\bar{c}(t)$ in order to construct such a family of neighboring extremals. The re-optimization procedure is treated in a straightforward manner in Ref. 1 and 2, but, since it is required that the standard end constraints be satisfied, a potential instability arises as $t_0 \to t_f$. The point of view to be adopted here will eliminate this difficulty by interpreting the minimization of $p = \bar{\nu}^T \bar{p}$ (Eq. 2) to be "equivalent" to the minimization (at the fixed time t_f) of the quadratic form

$$q = \alpha \bar{p}^T \bar{p} = \alpha \sum_{i=1}^{l} p_i^2 \tag{22}$$

where α is a positive constant of proportionality, and the $p_i(\bar{x},t)$ are the "boundary functions" defined above. (It is assumed that the boundary functions are suitably normalized to make them dimensionally compatible.) It will be seen that this approach leads to a simple and well-behaved control law, which is a generalization of the well-known "required velocity" concept currently being employed for guidance of rocket vehicles.

The first-order equivalence of the minimization of the penalty functions p and q can be established by supposing that a small initial condition error has been applied to the standard trajectory, and a perturbed control program has been constructed such that the penalty function q is minimized. From the discussion in Sec. III, it follows that

$$\bar{\eta}_q^T = 2\alpha \, \bar{p}_m^T \, [P \; U \; G] = 0 \tag{23}$$

where $\bar{\eta}_q^T$ is the influence vector corresponding to the penalty function q, and $\bar{p}_m$ is the minimized value of the boundary function vector

$$\bar{p}^T = (p_1, p_2, \cdots, p_l) \tag{24}$$

As the initial condition perturbation approaches zero, the elements of the influence matrix $[P \; U \; G]$, evaluated along the neighboring optimum trajectory, become equal to the values obtained on the standard trajectory. From Eq. (10) and (23) it follows that $\bar{\nu}$ and $\bar{p}_m$ are both perpendicular to the columns of $[P \; U \; G]$. Thus, if these column vectors span the space perpendicular to $\bar{\nu}$ and $\bar{p}_m$, it follows that $\bar{\nu}$ and $\bar{p}_m$ become parallel, i.e., there is a constant β such that

$$\bar{p}_m = \beta \bar{\nu} \tag{25}$$

Setting $(2\alpha\beta) = 1$, the influence functions $\bar{\eta}_q$ and $\bar{\eta}$ (and the differential corrections $\bar{\lambda}_q$ and $\bar{\lambda}$) become identical, and the two optimization problems can be said to be equivalent in the first-order sense in the limiting case $|\bar{p}_m| \to 0$. It is therefore reasonable to interpret the Lagrange multiplier vector on the standard trajectory to be the limiting value of

$$\bar{\nu}^T = \left(1, \; \frac{p_{m2}}{p_{m1}}, \; \cdots, \; \frac{p_{ml}}{p_{m1}} \right) \tag{26}$$

and to obtain a neighboring optimum control law which derives $\delta \bar{\nu}$ from Eq. (26). From the definition of q it can be seen that this approach to the neighboring optimum control problem yields a perturbed trajectory which attains the standard $\bar{p}$ "as closely as possible." A geometrical interpretation of this analysis is presented in Fig. 1.
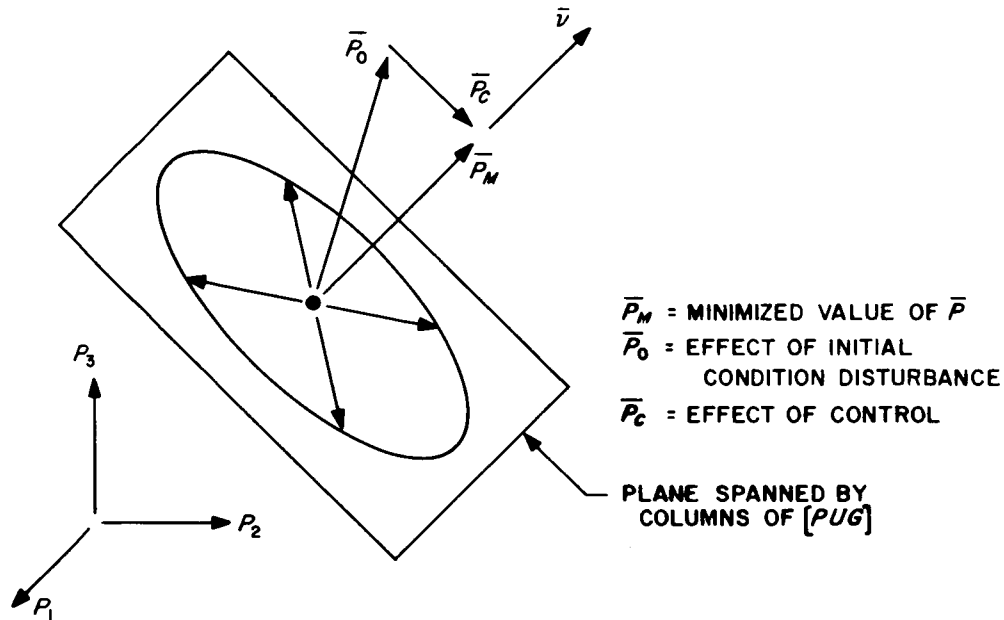

Fig. 1.   The Control Geometry

It will be convenient to define a rotated coordinate system (*), where

$$\bar{p}^{\,*} = \bar{q} = L\,\bar{p} \tag{27}$$

and hence

$$Q = \left[\frac{\partial \bar{q}}{\partial \bar{x}}\right]_{t_f} = L\left[\frac{\partial \bar{p}}{\partial \bar{x}}\right]_{t_f} = LP \tag{28}$$

where L is an orthogonal transformation chosen so that

$$(\bar{\nu}_s^*)^T = (L\,\bar{\nu}_s)^T = (k_s, 0, \cdots, 0) \tag{29}$$

From this construction it follows that the q_1 direction lies along $\bar{\nu}$, and that $k_s = |\bar{\nu}_s|$. It is always possible to construct a rotation matrix L when given $\bar{\nu}$, by a Gramm-Schmidt process, for example. (The L matrix is not unique.)

Consider now the variation in $\bar{\eta}(t)$ (Eq. 10) between the control reference trajectory and a neighboring extermal. Let it be assumed that the U and Q matrix elements are invariant, and that the mixed partials of the f_i with respect to c_j and x_k can be ignored. Then, dropping the (*) notation,

$$\delta\,\bar{\eta}^T(t) = 0 = \delta\,[(Q\,U\,C)^T\,\bar{\nu}\,]$$

$$= (Q\,U\,C)^T\,\delta\,\bar{\nu} + D\,\delta\,\bar{c} \tag{30}$$

where

$$\delta\bar{\nu}^T = \left[\,0,\ \left(\frac{\delta q_2}{q_1}\right), \cdots,\ \left(\frac{\delta q_l}{q_1}\right)\right] \tag{31}$$

and D is an $m \times m$ matrix with elements

$$d_{ij} = \sum_{k=1}^{n} \lambda_k \left(\frac{\partial^2 f_k}{\partial c_i\,\partial c_j}\right) \tag{32}$$

(The D matrix was discussed in Section III.) Let the variation in $\bar{q}$ due to state variable deviations at some initial time t_0 be

$$\delta\,\bar{q}_0 = QU\,(t_f,\,t_0)\,\delta\,\bar{x}\,(t_0) \tag{33}$$

and the variation in $\overline{q}$ after corrective control action be

$$\delta \overline{q}_f = \delta \overline{q}_0 - Q \int_{t_0}^{t_f} UG \, \delta \overline{c} \, dt$$

$$= \delta \overline{q}_0 - \int_{t_0}^{t_f} (QUG) \, (D^{-1}) \, (QUG)^T \, \delta \overline{\nu} \, dt \tag{34}$$

where Eq. (30) has been used to determine $\overline{c}(t)$. (The existence of the inverse of D, and the choice of the minus sign in Eq. (34), will be discussed in Section VI.) From Eq. (10), and because of the construction of $\overline{q}$, the first row of the matrix (QUG) is zero, and $(k_s + \delta q_1) = k$ can be directly evaluated from Eq. (33). Equation (34) can now be written

$$\delta \overline{q}_f = \delta \overline{q}_0 - \frac{1}{k} \, W(t_f, t_0) \, \delta \overline{q}_f \tag{35}$$

where

$$W(t_f, t_0) = \int_{t_0}^{t_f} (QUG) \, (D^{-1}) \, (QUG)^T \, dt \tag{36}$$

Thus

$$\delta \overline{q}_f = k \, [kI + W(t_f, t_0)]^{-1} \, \delta \overline{q}_0 \tag{37}$$

Note that Eq. (37) always guarantees the system response to be well behaved for arbitrary input $\delta \overline{q}_0$, even as $t_0 \to t_f$. The case $k = 0$ will be discussed in Section VI. Equation (37) yields the $\delta \overline{q}_f$ which has minimum magnitude, resulting from the given variation in initial conditions, $\delta \overline{x}(t_0)$, and control action between t_0 and t_f. The process can be thought of as a nonlinear feed-back loop, as shown in Fig. 2.
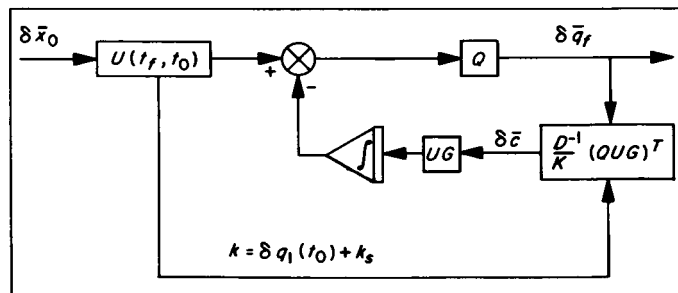

Fig. 2. The Control Process

V. THE VARIATION IN FINAL TIME

The above fixed-time analysis ignored the effect of changing the q function by varying the time of terminating the control accelerations by a small amount Δt. Let the "control time" t_c be $t_c = t_f + \Delta t$, where the magnitude of the control acceleration in the interval Δt is equal to its standard value at t_f if Δt is positive, and equal to zero if Δt is negative. Consider t_c to be the time at which the $\bar{q}$ is defined, then

$$
\begin{aligned}
\delta \bar{q}_c &= \bar{q}(t_c) - \bar{q}_s(t_c) \\
&= \left[\bar{q} + Q \, \dot{\bar{x}} \, \Delta t + \left(\frac{\partial \bar{q}}{\partial t_f} \right) \Delta t \right]_{t_f} - \left[\bar{q}_s + Q \, \dot{\bar{x}}_s \, \Delta t + \left(\frac{\partial \bar{q}}{\partial t_f} \right) \Delta t \right]_{t_f} = \delta \bar{q}_f + Q_v \Delta \bar{v} \quad (38)
\end{aligned}
$$

where Q_v is the partial derivative matrix relating $\bar{q}$ to the "velocity coordinates" $\bar{v}$, i.e., those coordinates with derivatives which are functions of $\bar{c}(t_f)$. Thus

$$
Q_v = \left[\frac{\partial \bar{q}}{\partial \bar{v}} \right] \quad (39)
$$

The $\Delta \bar{v}$ is the increment in $\bar{v}$ due to changing the control acceleration vector in Δt, thus

$$
\Delta \bar{v} = \pm \left[\bar{f}(\bar{x}_{sf}, \bar{c}_{sf} + \delta \bar{c}_f, t_f) - \bar{f}(\bar{x}_{sf}, 0, t_f) \right] \Delta t \quad (40)
$$

[Note that in the definition of $\Delta \bar{v}$ the effect of small changes in state variables due to terms of the form $(\delta \dot{\bar{x}})(\Delta t)$ have been ignored, which is consistent with this first order theory.] It is easily verified that the $\Delta \bar{v}$ which minimizes $q_c = (\bar{q}_c^T \bar{q}_c)$ is

$$
\Delta \bar{v} = - [Q_v^T Q_v]^{-1} Q_v^T \delta \bar{q}_f \quad (41)
$$

where it is assumed that the indicated inverse exists (the case where it does not will be discussed in the following section). The quantity $\Delta \bar{v}$ will be called the "generalized velocity-to-be-gained", in accordance with the analogous concept in current guidance and control literature. It is the delta function of velocity which must be added at t_f in order to satisfy the constraints in a least squares sense. It is attained by varying $\bar{c}(t_f)$ and Δt. The resultant $\delta \bar{q}$ is

$$\delta \overline{q}_c = \{I - Q_v [Q_v^T Q_v]^{-1} Q_v^T\} \delta \overline{q}_f \tag{42}$$

Notice that if t_c is considered as the final time, Eq. (42) yields

$$\overline{q}_c^T Q_v = (\lambda_1, \lambda_2, \lambda_3)_{t_c} = 0 \tag{43}$$

This is the "stopping condition" which terminates the control.

VI.   CONTROLLABILITY

The system will be said to be controllable with respect to a constraint q_i if $q_i(t_c)$ is zero for all initial condition errors $\delta \overline{x}(t_0)$ (Ref. 8). From Eq. (42) it immediately follows that all q_i are controllable for small changes in t_c if and only if $(Q_v)^{-1}$ exists. If this is not the case, it is necessary to consider $\overline{q}(t_f)$ by referring to Eq. (37). It can be assumed that the inverse of $(Q_v^T Q_v)$ exists, for this matrix can always be diagonalized, and the nonsingular submatrix corresponding to the nonzero eigenvalues can then be considered. Thus if

$$M^T [Q_v^T Q_v] \ M = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \tag{44}$$

where M is an orthogonal matrix and Λ is diagonal, then

$$[(M^T Q_v^T)(Q_v M)]^{-1} = \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{45}$$

would be defined as the "generalized inverse" (see work by Penrose).

The controllability of the system is related to properties of the W matrix. From Eq. (36) it can be seen that W is symmetric and positive semi-definite, since D has these properties. The symmetry of D follows from Eq. (32), and the positive semi-definite property follows from the discussion in Section III. (W would be negative semi-definite if p were maximized.) The case $|D| = 0$ over any interval along the path is a singularity corresponding to an inflection point in ordinary minima problem. On such an interval D^{-1} does not exist, and the integrand in Eq. (36) must be set equal to zero. With this discussion in mind, imagine the $W(t_f, t_0)$ matrix to be diagonalized by an orthogonal transformation. Denoting the transformed system by Λ, Eq. (37) becomes

$$\hat{\overline{q}} \, (t_f) \, = \, H \, \hat{\overline{q}} \, (t_0)$$

(46)

where H is a diagonal matrix with elements

$$h_{ii} \; = \; \begin{cases} 1 \text{ for } i = 1 \\[2mm] 1 \text{ if } \hat{w}_{ii} = 0 \\[4mm] \dfrac{k}{k + \hat{w}_{ii}} \text{ if } \hat{w}_{ii} \neq 0 \end{cases}$$

(47)

The $\hat{w}_{ii}$ are the always-positive eigenvalues of the $\hat{W}$ matrix. It is Eq. (47) which demands that $(-W)$ instead of $(+W)$ be employed in Eq. (34) in order to minimize δq. (The sign would be reversed if p were maximized instead of minimized on the standard trajectory.) Note that the number of nonzero eigenvalues is equal to the rank of Q or the rank of G, whichever is smaller.

It is now possible to say that the system is "in the limit" controllable with respect to the components of $\hat{\overline{q}}$ corresponding to the nonzero eigenvalues of $\hat{W} \, (t_f, \, t_0)$, for

$$\begin{cases} \hat{w}_{ii} \, (t_f, \, t_0) \to \infty \\[2mm] \text{and/or } k \to 0 \end{cases}$$

implies $k/(k + w_{ii}) \to 0$. Intuitively, this means that these components are approximately nulled at t_f, and the final control lies nearly along the q_1 direction, thereby nulling q_1.

VII. THE MINIMUM TIME TRAJECTORY – THE MINIMUM VELOCITY-TO-BE-GAINED

Suppose there is some vector to be nulled "as closely as possible", and, following Eq. (41), the velocity-to-be-gained at t_f is

$$\Delta \overline{v} \, = \, - \, [Q_v^T Q_v]^{-1} \, Q_v^T Q \, \delta \overline{x}_f$$

(48)

But, from Eq. (40),

$$\Delta \overline{v}^T \, \Delta \overline{v} \, = \, \text{(constant) } \Delta t^2$$

(49)

Thus, minimizing the magnitude of $\Delta \bar{v}$ minimizes the variation in final time, $\Delta t = t_c - t_f$. Letting $\bar{p} = \Delta \bar{v}$, and hence $P = - [Q_v^T Q_v]^{-1} Q_v^T Q$, the "minimum time" trajectory results.

This case presents an interpretation of the "required velocity" approach to present day rocket guidance problems, where at most three guidance equations, g_i, are to be nulled at t_f by the application of $\Delta \bar{v}$, the g_i defined such that the inverse of $[\partial \bar{g}/\partial \bar{v}]$ exists. Eq. (48) then becomes

$$\Delta \bar{v}. = - \left[\frac{\partial \bar{g}}{\partial \bar{v}} \right]^{-1} \left[\frac{\partial \bar{g}}{\partial \bar{x}} \right] \delta \bar{x}_f \tag{50}$$

The conventional treatment of this problem usually involves constructing a steering program which causes the vehicle to stay near the standard trajectory during the early portions of flight, and follows Eq. (50) near t_f. The above analysis provides a technique for accomplishing the desired task "optimally", eliminating the ever-present problem of causing all components of $\Delta \bar{v}$ to go to zero simultaneously.

VIII.   AN EXAMPLE [4]

Suppose a free-fall trajectory is to be attained with maximum horizontal speed, $\dot{x}$, $(-\dot{x}$ is a minimum) subject to the constraint $\dot{y} = 0$ and $y = y_s$, where the coordinate system is as shown in Fig. 3. The motion will be assumed planar in a constant gravity field, g, with constant thrust acceleration, a, according to

$$\ddot{x} = a \cos \theta \tag{51}$$
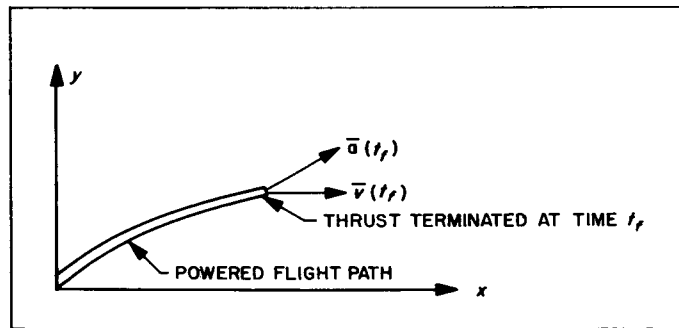
$$\ddot{y} = a \sin \theta - g \tag{52}$$

Fig. 3.   The Standard Trajectory

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[4] Discussed in Ref. 1.

The control variable is θ. Equation (1) becomes

$$
\frac{d}{dt}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
=
\begin{bmatrix}
a \cos \theta \\
a \sin \theta - g \\
x_1 \\
x_2
\end{bmatrix}
\tag{53}
$$

The U matrix (Eq. 2 through 7) is

$$
U(t_f, t) =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\tau & 0 & 1 & 0 \\
0 & \tau & 0 & 1
\end{bmatrix}
\tag{54}
$$

where $\tau = (t_f - t)$. Let the normalized $\bar{p}$ be

$$
\bar{p}^T = \left(\frac{\dot{x} - v}{v} , \frac{\dot{y}}{v} , \frac{y - r}{r} \right)
\tag{55}
$$

where $v = \dot{x}_s(t_f)$ and $r = y_s(t_f)$. Then

$$
P = \left[\frac{\partial \bar{p}}{\partial \bar{x}} \right]_{t_f} =
\begin{bmatrix}
\dfrac{1}{v} & 0 & 0 & 0 \\
0 & \dfrac{1}{v} & 0 & 0 \\
0 & 0 & 0 & \dfrac{1}{r}
\end{bmatrix}
\tag{56}
$$

From Eq. (53)

$$
G = a
\begin{bmatrix}
-\sin \theta \\
\cos \theta \\
0 \\
0
\end{bmatrix}
\tag{57}
$$

Suppose $\overline{\nu}_s^T = (1, 0, \sigma)$, where σ is known. Then, from Eq. (24),

$$\tan \theta_s(t) = \frac{(v \sigma \tau)}{r} \tag{58}$$

Take the transformation

$$L = \left(\frac{1}{\sqrt{1 + \sigma^2}} \right) \begin{bmatrix} 1 & 0 & \sigma \\ 0 & \sqrt{1 + \sigma^2} & 0 \\ \sigma & 0 & -1 \end{bmatrix} \tag{59}$$

so that

$$L\overline{\nu}_s = \begin{pmatrix} \sqrt{1 + \sigma^2} \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

Then

$$Q = LP = \left(\frac{1}{\sqrt{1 + \sigma^2}} \right) \begin{bmatrix} \dfrac{1}{v} & 0 & 0 & \dfrac{\sigma}{r} \\ 0 & \dfrac{\sqrt{1 + \sigma^2}}{v} & 0 & 0 \\ \dfrac{\sigma}{v} & 0 & 0 & \dfrac{-1}{r} \end{bmatrix} \tag{61}$$

$$(QUG) = a \cos \theta \begin{bmatrix} 0 \\ \dfrac{1}{v} \\ \dfrac{-\tau \nu}{r} \end{bmatrix} \quad \text{where } \nu = \sqrt{1 + \sigma^2} \tag{62}$$

$$D = a \frac{\partial}{\partial \theta} \left(-\frac{\sin \theta}{v} + \frac{\sigma \tau \cos \theta}{r} \right) = \frac{-a}{v \cos \theta} \tag{63}$$

(Notice that D is negative definite for θ in the first or fourth quadrant, which is guaranteed by Eq. (58).) Since, by Eq. (58),

$$\frac{d\theta}{dt} = -\left(\frac{v\sigma}{r}\right) \cos^2 \theta \tag{64}$$

Eq. (36) becomes

$$W(t_f, t_0) = -\left(\frac{ar}{v^2 \sigma}\right) \int_{\theta(t_0)}^{\theta(t_f) = 0} d\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos \theta & -\left(\frac{\nu}{\sigma}\right) \sin \theta \\ 0 & -\left(\frac{\nu}{\sigma}\right) \sin \theta & \left(\frac{\nu}{\sigma}\right)^2 \frac{\sin^2 \theta}{\cos \theta} \end{bmatrix}$$

where the minus sign has been chosen because of the negative-definite property of D. Applying Eq. (37),

$$\delta q_1 = \delta q_{10} \tag{65}$$

$$\begin{pmatrix} \delta q_2 \\ \delta q_3 \end{pmatrix}_{t_f} = \rho \begin{bmatrix} k + w_{33} & -w_{21} \\ -w_{21} & k + w_{22} \end{bmatrix} \begin{bmatrix} \delta q_{20} \\ \delta q_{30} \end{bmatrix} \tag{66}$$

where

$$\rho = \frac{k}{(k + w_{22})(k + w_{33}) - w_{21}^2} \tag{67}$$

Letting $k = \nu$, the matrix elements in Eq. (66) may be interpreted as impulse response functions which relate the controlled response δq_{if} to the "open-loop" response δq_{i0}. These are plotted in Fig. 4 and 5 for $a = 96.0$ ft/sec^2, $v = 25{,}000$ ft/sec, $r = 600{,}000$ ft, and $\sigma = 0.1$.

The effect of varying the final time can be obtained from Eq. (42).
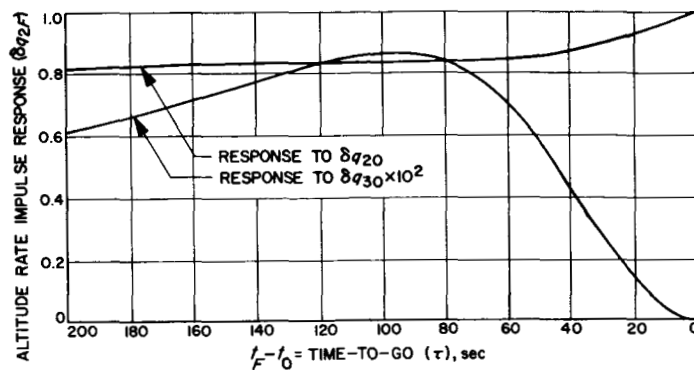
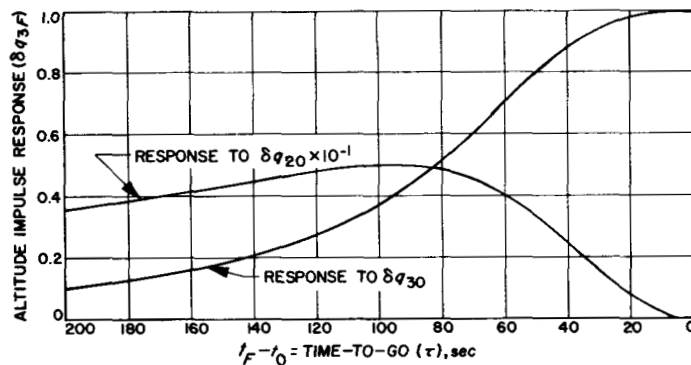Fig. 4. Altitude Rate Impulse Response vs Time

Fig. 5. Altitude Impulse Response vs Time

IX. CONCLUSION

This paper has presented an analysis of the final value control problem which provides a simple scheme for applying the control acceleration. It has been shown that the control functions and system response are always well behaved, causing the standard end conditions to be attained "as closely as possible". The essence of the technique is the interpretation of the problem of minimizing a function of the end coordinates, subject to constraints, to be equivalent to the minimization of a certain unconstrained quadratic form. This approach can provide some insight into the properties of neighboring optimum trajectories, and perhaps suggest a search routine for generating standard trajectories. Certain approximations suggest themselves, such as setting $k = |\bar{\nu}|$ (Eq. 37) for all perturbed trajectories, thereby obtaining a linear control (Fig. 2). The problem of estimating the initial state of the system from noise contaminated data fits nicely within the framework of this analysis. This area is yet to be explored.

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